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PRESSURE METHOD FOR THE NUMERICAL SOLUTION OF TRANSIENT, COMPRESSIBLE FLUID FLOWS

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SUMMARY

In this paper the *pressure method* for incompressible fluid flow simulation is extended and applied to the numerical simulation of compressible *fluid flow*. The governing equations, obtained from the physical principles of conservation of momentum, mass and energy, are first studied from a characteristic point of view. Then they are discretized with a semi-implicit finite difference technique in such a fashion that stability is achieved independently of the speed of sound. The resulting algorithm is fast, accurate and particularly efficient in subsonic flow calculations. As an example, the computer simulation of the von Kármán vortex street is described and discussed.

KEY WORDS Compressible Fluid Flow Numerical Analysis Pressure Method von Kármán Vortex Street

INTRODUCTION

In recent years a special pressure method has been used successfully to simulate numerically a variety of incompressible fluid flow problems.[1–5] This method incorporates a space-staggered mesh to discretize Navier–Stokes types of equations, and the pressure, at each time step, is obtained implicitly by solving a simple system of linear algebraic equations. The resulting velocity field is exactly discrete divergence free, in agreement with the assumption of fluid incompressibility. The method applies to transient flow simulation of non-homogeneous, non-isothermal, free surface and porous reservoir problems in two and in three space dimensions (see Reference 1 for details).

This method for incompressible fluids (in which the speed of sound is assumed to be infinite) is extended here to compressible fluids, in which the speed of sound is a function of the pressure and of the fluid density. Again, a space staggered mesh is used and the pressure, at each time step, is determined implicitly in such a way that the equation of state is satisfied identically. Of course, a fully implicit discretization of the governing equations would lead to a method which is unconditionally stable; this procedure, however, involves the simultaneous solution of a large number of coupled non-linear difference equations. For efficiency, then, only some terms of the governing equations are discretized implicitly. Specifically, since in subsonic flows the time scale of the speed of sound is faster than the time scale of the flow, our method uses an implicit discretization only for those terms which are related to the speed of sound. As a result the method is unconditionally stable with respect to the Courant condition for the speed of sound, which is the

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most restrictive condition for fully explicit methods in subsonic flow calculations. For supersonic or slightly subsonic problems, a fully explicit or a fully implicit method becomes more efficient because the Courant stability condition on the speed of sound is no longer the critical restriction. A similar idea of semi-implicit discretization of the governing equations was used in the SOLA-ICE algorithm,[6] which allows for a more general equation of state, but requires, at each time step, the iterative solution of a non-linear system of equations for the pressure. Moreover, the pressure computed with the SOLA-ICE algorithm does not rigorously satisfy the equation of state. Our method requires, at each time step, the solution of a simple linear system of algebraic equations. We will show that this system has a unique solution and can be solved quickly and economically by successive overrelaxation iteration. Alternatively, the use of a fully explicit time splitting method which uses small time steps only for the fast time scale also results in an efficient algorithm. This approach, recently investigated by Le Veque and Oliger,[7] becomes impractical if the speed of sound is too large because the smaller time step has to satisfy the Courant condition for the speed of sound.

GOVERNING EQUATIONS

The basic equations of gas dynamics which are commonly used to model transient compressible flow in a given domain are derived from the physical principles of conservation of linear momentum, mass and energy (see Reference 8 for details). The equations we will use are the following: Euler's equation of motion

$$\rho\left[\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q}\cdot\nabla)\mathbf{q}\right] = -\nabla p \tag{1}$$

where ρ is the fluid density, $\mathbf{q} = (u, v, w)^{\mathrm{T}}$ is the fluid velocity, and p is the pressure; the continuity equation

$$\frac{\partial \rho}{\partial t} + (\mathbf{q}\cdot\nabla)\rho = -\rho\nabla\cdot\mathbf{q} \tag{2}$$

and the energy equation

$$\rho\left[\frac{\partial I}{\partial t} + (\mathbf{q}\cdot\nabla)I\right] = -p\nabla\cdot\mathbf{q} \tag{3}$$

where I is the internal energy per unit mass.

The equation of state relating the pressure to both the density and the internal energy, for an ideal gas, is taken to be

$$p = (\gamma - 1)\rho I \tag{4}$$

where the constant γ is the ratio of specific heats and is greater than unity. Note that viscosity, mass diffusion and thermal diffusion terms have not been included in equations (1)–(3). However, there is nothing in the general discussion which follows that limits us to consider inviscid flows only, and, at present, such terms have been omitted merely for simplicity.

Let us now define the internal energy E per unit of volume as $E = \rho I$. An equation for E is obtained by combining (2) with (3):

$$\frac{\partial E}{\partial t} + (\mathbf{q}\cdot\nabla)E = -\gamma E\nabla\cdot\mathbf{q} \tag{5}$$

Equation (5) will take the place of the energy equation (3), so that the mathematical system (1), (2)

and (5), written out in full in two dimensions, is

$$\left.\begin{aligned} \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}&=-\frac{1}{\rho}\frac{\partial p}{\partial x}\\ \frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}&=-\frac{1}{\rho}\frac{\partial p}{\partial y}\\ \frac{\partial E}{\partial t}+u\frac{\partial E}{\partial x}+v\frac{\partial E}{\partial y}&=-\gamma E\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\\ \frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial y}&=-\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) \end{aligned}\right\} \tag{6}$$

and the equation of state (4) reduces to

$$p=(\gamma-1)E \tag{7}$$

If the energy E in the third equation of system (6) is replaced with $p/(\gamma-1)$, system (6) can be written in matrix notation as

$$\frac{\partial \mathbf{W}}{\partial t}+A\frac{\partial \mathbf{W}}{\partial x}+B\frac{\partial \mathbf{W}}{\partial y}=0 \tag{8}$$

where $\mathbf{W}=(u,v,p,\rho)^{\mathrm{T}}$, and

$$A=\begin{bmatrix} u & 0 & \frac{1}{\rho} & 0\\ 0 & u & 0 & 0\\ \gamma p & 0 & u & 0\\ \rho & 0 & 0 & u \end{bmatrix},\quad B=\begin{bmatrix} v & 0 & 0 & 0\\ 0 & v & \frac{1}{\rho} & 0\\ 0 & \gamma p & v & 0\\ 0 & \rho & 0 & v \end{bmatrix}$$

If I denotes the identity matrix, the characteristic equation of system (8) is

$$\det(qI+rA+sB)=0 \tag{9}$$

that is,

$$(q+ru+sv)^2[(q+ru+sv)^2-a^2(r^2+s^2)]=0 \tag{10}$$

where $a^2=\gamma p/\rho$, and a is the local speed of sound. The triples (q,r,s) satisfying equation (10) are the directions normal to the characteristic cone at its vertex.[9] Equation (10) decomposes into the following two equations:

$$(q+ru+sv)^2=0 \tag{11}$$

and

$$(q+ru+sv)^2-a^2(r^2+s^2)=0 \tag{12}$$

Hence, the characteristic cone consists of the line through the vertex parallel to the vector $(1,u,v)$ and the right cone whose axis is parallel to $(1,u,v)$ with generating angle α, where $a=\tan\alpha$ (see Reference 9, p. 385). Note that, whereas the first part of the characteristic cone depends only on the fluid velocity u and v, the second part, which is defined by equation (12), depends also upon the speed of sound. Note also that the speed of sound in equation (10) arises from the coefficients of the derivatives on the right-hand sides of the first three equations of system (6). Thus, these derivatives *must* be discretized implicitly in order for a numerical method to be unconditionally stable with respect to the Courant condition for the speed of sound. If the left-hand sides of each equation in

system (6) are discretized with explicit finite differences, then a Courant type of stability condition depending on the fluid speed must be satisfied; the latter, however, is far less restrictive than the previous one in subsonic flow simulation. Based on this idea, we derive in the next section a finite difference scheme for system (6) which uses an implicit discretization only for the derivatives on the right-hand sides of the first three equations in system (6).

FINITE DIFFERENCE DISCRETIZATION

The finite difference mesh that we are going to use to discretize equations (6) consists of rectangular cells of width Δx and height Δy. The field variables u, v, p, E and ρ are defined at the locations shown in Figure 1: u-velocity at the centre of each vertical side of a cell; v-velocity at the centre of each horizontal side; pressure p, energy E and density ρ at each cell centre.

	$v_{i,j+1/2}$	
$u_{i-1/2,j}$	$p_{i,j}; E_{i,j}; \rho_{i,j}$	$u_{i+1/2,j}$
	$v_{i,j-1/2}$	

Figure 1. Position of field variables

Once a finite time step Δt has been chosen, the first two equations of system (6), that is, the momentum equations are discretized in the following way:

$$\left.\begin{aligned} \frac{u^{n+1}_{i+1/2,j} - u^n_{i+1/2,j}}{\Delta t} &= -F^n_{i+1/2,j} - \frac{1}{\rho^n_{i+1/2,j}} \frac{p^{n+1}_{i+1,j} - p^{n+1}_{i,j}}{\Delta x} \\ \frac{v^{n+1}_{i,j+1/2} - v^n_{i,j+1/2}}{\Delta t} &= -G^n_{i,j+1/2} - \frac{1}{\rho^n_{i,j+1/2}} \frac{p^{n+1}_{i,j+1} - p^{n+1}_{i,j}}{\Delta y} \end{aligned}\right\} \tag{13}$$

where $F^n_{i+1/2,j}$ and $G^n_{i,j+1/2}$ contain the finite differences corresponding to the spatial discretization of the convective terms at the time step t_n. Particular forms for $F^n_{i+1/2,j}$ and $G^n_{i,j+1/2}$ can be chosen in a variety of ways, as, for example, by using a first-order accurate upwind formula or a simple space centred difference formula. The latter, however, is unstable if no artificial viscosity is added to the system. A still more sophisticated formula can be obtained by using a higher order upwind differencing for convection.[10] For the time being $F^n_{i+1/2,j}$ and $G^n_{i,j+1/2}$ will be left unspecified. The discrete variables $\rho^n_{i+1/2,j}$ and $\rho^n_{i,j+1/2}$ in (13) are defined as simple averages from the closest scalar grid points.

The third equation of system (6), that is the energy equation, is discretized at each cell centre as follows:

$$\frac{E^{n+1}_{i,j} - E^n_{i,j}}{\Delta t} = -H^n_{i,j} - \gamma E^n_{i,j}\left(\frac{u^{n+1}_{i+1/2,j} - u^{n+1}_{i-1/2,j}}{\Delta x} + \frac{v^{n+1}_{i,j+1/2} - v^{n+1}_{i,j-1/2}}{\Delta y}\right) \tag{14}$$

where $H^n_{i,j}$ contains finite differences corresponding to the spatial discretization of the convective terms of the energy equation. A particular form for $H^n_{i,j}$ can also be chosen in a variety of ways and, for the time being, it will be left unspecified.

The last equation of system (6), which can be written in the following conservative form:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \tag{15}$$

has been studied in several discretized forms. One such form, which is conservative, is the particular upwind differencing scheme defined as follows:

$$\frac{\rho_{i,j}^{n+1}-\rho_{i,j}^{n}}{\Delta t}+\frac{\rho_{i+1/2,j}^{n}u_{i+1/2,j}^{n}-\rho_{i-1/2,j}^{n}u_{i-1/2,j}^{n}}{\Delta x}+\frac{\rho_{i,j+1/2}^{n}v_{i,j+1/2}^{n}-\rho_{i,j-1/2}^{n}v_{i,j-1/2}^{n}}{\Delta y}=0 \tag{16}$$

where

$$\begin{aligned}
&\rho_{i+1/2,j}^{n}=\rho_{i,j}^{n} \quad \text{for} \quad u_{i+1/2,j}^{n}>0, \quad \rho_{i+1/2,j}^{n}=\rho_{i+1,j}^{n} \quad \text{for} \quad u_{i+1/2,j}^{n}<0\\
&\rho_{i-1/2,j}^{n}=\rho_{i-1,j}^{n} \quad \text{for} \quad u_{i-1/2,j}^{n}>0, \quad \rho_{i-1/2,j}^{n}=\rho_{i,j}^{n} \quad \text{for} \quad u_{i-1/2,j}^{n}<0\\
&\rho_{i,j+1/2}^{n}=\rho_{i,j}^{n} \quad \text{for} \quad v_{i,j+1/2}^{n}>0, \quad \rho_{i,j+1/2}^{n}=\rho_{i,j+1}^{n} \quad \text{for} \quad v_{i,j+1/2}^{n}<0\\
&\rho_{i,j-1/2}^{n}=\rho_{i,j-1}^{n} \quad \text{for} \quad v_{i,j-1/2}^{n}>0, \quad \rho_{i,j-1/2}^{n}=\rho_{i,j}^{n} \quad \text{for} \quad v_{i,j-1/2}^{n}<0
\end{aligned}$$

Finally, the equation of state (7) is required to be satisfied at each cell center at all time levels:

$$p_{i,j}^{n+1}=(\gamma-1)E_{i,j}^{n+1} \tag{17}$$

Once a discretization formula for the convective terms in the momentum and in the energy equation has been chosen, $F_{i+1/2,j}^{n}$, $G_{i,j+1/2}^{n}$ and $H_{i,j}^{n}$ are determined and the finite difference equations (13), (14), (16) and (17) can be used to generate recursively values of u, v, E, ρ and p at time level $n+1$ from known values at time level $n, n=0,1,\ldots$. Specifically, values for $\rho_{i,j}^{n+1}$ can be determined explicitly in each cell by using equation (16), while values for $u_{i+1/2,j}^{n+1}$, $v_{i,j+1/2}^{n+1}$, $E_{i,j}^{n+1}$ and $p_{i,j}^{n+1}$ follow by solving the *linear* system of equations determined by (13), (14) and (17), over the cell configuration. Since this system has to be solved at each time step and because most of the computational effort of the method is taken by the solution of this system, the next section will be devoted to discussing the special iterative technique to be used.

PRESSURE EVALUATION

Unlike implicit methods, the determination of u^{n+1}, v^{n+1}, E^{n+1} and p^{n+1} can be accomplished efficiently because the corresponding linear system determined by equations (13), (14) and (17) can be reduced to a smaller one having only the pressures $p_{i,j}^{n+1}$ as unknowns. The reduced system is then solved with the successive overrelaxation iterative method, which can be implemented in such a fashion that the unknown pressures $p_{i,j}^{n+1}$ and velocities $u_{i+1/2,j}^{n+1}$ and $v_{i,j+1/2}^{n+1}$ are determined simultaneously. Finally, the new energy $E_{i,j}^{n+1}$ satisfying (14) can be evaluated directly from the equation of state (17). Specifically, we proceed as follows:

Elimination of $E_{i,j}^{n+1}$ from (14) and (17) yields

$$p_{i,j}^{n+1}=(\gamma-1)(E_{i,j}^{n}-\Delta t H_{i,j}^{n})-\gamma(\gamma-1)E_{i,j}^{n}\Delta t\left(\frac{u_{i+1/2,j}^{n+1}-u_{i-1/2,j}^{n+1}}{\Delta x}+\frac{v_{i,j+1/2}^{n+1}-v_{i,j-1/2}^{n+1}}{\Delta y}\right) \tag{18}$$

If now $u_{i+1/2,j}^{n+1}$, $u_{i-1/2,j}^{n+1}$, $v_{i,j+1/2}^{n+1}$ and $v_{i,j-1/2}^{n+1}$ in (18) are replaced with the corresponding expressions that one obtains from (13), we find the following finite difference equation for the pressure:

$$\begin{aligned}
&p_{i,j}^{n+1}-\gamma(\gamma-1)E_{i,j}^{n}(\Delta t)^2\left[\frac{p_{i+1,j}^{n+1}-p_{i,j}^{n+1}}{\rho_{i+1/2,j}^{n}(\Delta x)^2}-\frac{p_{i,j}^{n+1}-p_{i-1,j}^{n+1}}{\rho_{i-1/2,j}^{n}(\Delta x)^2}+\frac{p_{i,j+1}^{n+1}-p_{i,j}^{n+1}}{\rho_{i,j+1/2}^{n}(\Delta y)^2}-\frac{p_{i,j}^{n+1}-p_{i,j-1}^{n+1}}{\rho_{i,j-1/2}^{n}(\Delta y)^2}\right]\\
&=(\gamma-1)(E_{i,j}^{n}-\Delta t H_{i,j}^{n})-\gamma(\gamma-1)E_{i,j}^{n}\Delta t\left[\frac{u_{i+1/2,j}^{n}-u_{i-1/2,j}^{n}}{\Delta x}+\frac{v_{i,j+1/2}^{n}-v_{i,j-1/2}^{n}}{\Delta y}\right]\\
&\quad+\gamma(\gamma-1)E_{i,j}^{n}(\Delta t)^2\left(\frac{F_{i+1/2,j}^{n}-F_{i-1/2,j}^{n}}{\Delta x}+\frac{G_{i,j+1/2}^{n}-G_{i,j-1/2}^{n}}{\Delta y}\right)
\end{aligned} \tag{19}$$

Equation (19), applied on each finite difference cell, yields a system of linear equations with unknowns $p_{i,j}^{n+1}$. This system, since $\gamma > 1$ and $\rho_{i,j}^n$ and $E_{i,j}^n$ are positive quantities, is strictly diagonally dominant and has a unique solution which can be obtained by successive overrelaxation.

In sweeping the cells of the computational domain from left to right and bottom to top, the successive overrelaxation formula for $p_{i,j}^{n+1}$ in (19) is given by

$$\begin{aligned}(p_{i,j}^{n+1})^{(v+1)} = (p_{i,j}^{n+1})^{(v)} - \frac{\omega}{\beta_{i,j}^n}\Bigg\{&(p_{i,j}^{n+1})^{(v)} - \gamma(\gamma-1)E_{i,j}^n(\Delta t)^2 \\ &\times\left[\frac{(p_{i+1,j}^{n+1})^{(v)} - (p_{i,j}^{n+1})^{(v)}}{\rho_{i+1/2,j}^n(\Delta x)^2} - \frac{(p_{i,j}^{n+1})^{(v)} - (p_{i-1,j}^{n+1})^{(v+1)}}{\rho_{i-1/2,j}^n(\Delta x)^2}\right. \\ &\left.+\frac{(p_{i,j+1}^{n+1})^{(v)} - (p_{i,j}^{n+1})^{(v)}}{\rho_{i,j+1/2}^n(\Delta y)^2} - \frac{(p_{i,j}^{n+1})^{(v)} - (p_{i,j-1}^{n+1})^{(v+1)}}{\rho_{i,j-1/2}^n(\Delta y)^2}\right] \\ &-(\gamma-1)[E_{i,j}^n - (\Delta t)H_{i,j}^n] + \gamma(\gamma-1)E_{i,j}^n\Delta t\left(\frac{u_{i+1/2,j}^n - u_{i-1/2,j}^n}{\Delta x} + \frac{v_{i,j+1/2}^n - v_{i,j-1/2}^n}{\Delta y}\right) \\ &-\gamma(\gamma-1)E_{i,j}^n(\Delta t)^2\left(\frac{F_{i+1/2,j}^n - F_{i-1/2,j}^n}{\Delta x} + \frac{G_{i,j+1/2}^n - G_{i,j-1/2}^n}{\Delta y}\right)\Bigg\}\end{aligned} \tag{20}$$

where v is the iteration index, ω is an overrelaxation factor, and $\beta_{i,j}^n$ is defined as

$$\beta_{i,j}^n = 1 + \gamma(\gamma-1)E_{i,j}^n(\Delta t)^2\left[\frac{1}{(\Delta x)^2}\left(\frac{1}{\rho_{i+1/2,j}^n} + \frac{1}{\rho_{i-1/2,j}^n}\right) + \frac{1}{(\Delta y)^2}\left(\frac{1}{\rho_{i,j+1/2}^n} + \frac{1}{\rho_{i,j-1/2}^n}\right)\right] \tag{21}$$

Now, in cell (i,j), define the following iterative scheme for equations (13):

$$\left.\begin{aligned}(u_{i+1/2,j}^{n+1})^{(v)} &= u_{i+1/2,j}^n - (\Delta t)F_{i+1/2,j}^n - \frac{\Delta t}{\rho_{i+1/2,j}^n}\frac{(p_{i+1,j}^{n+1})^{(v)} - (p_{i,j}^{n+1})^{(v)}}{\Delta x} \\ (u_{i-1/2,j}^{n+1})^{(v+1/2)} &= u_{i-1/2,j}^n - (\Delta t)F_{i-1/2,j}^n - \frac{\Delta t}{\rho_{i-1/2,j}^n}\frac{(p_{i,j}^{n+1})^{(v)} - (p_{i-1,j}^{n+1})^{(v+1)}}{\Delta x} \\ (v_{i,j+1/2}^{n+1})^{(v)} &= v_{i,j+1/2}^n - (\Delta t)G_{i,j+1/2}^n - \frac{\Delta t}{\rho_{i,j+1/2}^n}\frac{(p_{i,j+1}^{n+1})^{(v)} - (p_{i,j}^{n+1})^{(v)}}{\Delta y} \\ (v_{i,j-1/2}^{n+1})^{(v+1/2)} &= v_{i,j-1/2}^n - (\Delta t)G_{i,j-1/2}^n - \frac{\Delta t}{\rho_{i,j-1/2}^n}\frac{(p_{i,j}^{n+1})^{(v)} - (p_{i,j-1}^{n+1})^{(v+1)}}{\Delta y}\end{aligned}\right\} \tag{22}$$

so that the iterative formula (20) can also be written in the following way:

$$\begin{aligned}(p_{i,j}^{n+1})^{(v+1)} = (p_{i,j}^{n+1})^{(v)} - \frac{\omega}{\beta_{i,j}^n}\Bigg\{&(p_{i,j}^{n+1})^{(v)} - (\gamma-1)[E_{i,j}^n - (\Delta t)H_{i,j}^n] \\ &+(\gamma-1)\gamma E_{i,j}^n\Delta t\left[\frac{(u_{i+1/2,j}^{n+1})^{(v)} - (u_{i-1/2,j}^{n+1})^{(v+1/2)}}{\Delta x}\right. \\ &\left.+\frac{(v_{i,j+1/2}^{n+1})^{(v)} - (v_{i,j-1/2}^{n+1})^{(v+1/2)}}{\Delta y}\right]\Bigg\}\end{aligned} \tag{23}$$

The iterative formulae (22), (23), which are equivalent to (20), can be further simplified by defining the pressure change $(\delta p_{i,j}^{n+1})^{(\nu)}$ as

$$(\delta p_{i,j}^{n+1})^{(\nu)} = -\frac{\omega}{\beta_{i,j}^n}\left\{(p_{i,j}^{n+1})^{(\nu)} - (\gamma-1)[E_{i,j}^n - (\Delta t)H_{i,j}^n]\right.$$
$$+(\gamma-1)\gamma E_{i,j}^n \Delta t\left[\frac{(u_{i+1/2,j}^{n+1})^{(\nu)} - (u_{i-1/2,j}^{n+1})^{(\nu+1/2)}}{\Delta x}\right.$$
$$\left.\left.+\frac{(v_{i,j+1/2}^{n+1})^{(\nu)} - (v_{i,j-1/2}^{n+1})^{(\nu+1/2)}}{\Delta y}\right]\right\} \tag{24}$$

so that (23) reduces to

$$(p_{i,j}^{n+1})^{(\nu+1)} = (p_{i,j}^{n+1})^{(\nu)} + (\delta p_{i,j}^{n+1})^{(\nu)} \tag{25}$$

and the velocity iterates on each side of cell (i,j) can be updated as follows:

$$\left.\begin{aligned}
(u_{i+1/2,j}^{n+1})^{(\nu+1/2)} &= (u_{i+1/2,j}^{n+1})^{(\nu)} + \frac{\Delta t}{\rho_{i+1/2,j}^n}\frac{(\delta p_{i,j}^{n+1})^{(\nu)}}{\Delta x}\\
(u_{i-1/2,j}^{n+1})^{(\nu+1)} &= (u_{i-1/2,j}^{n+1})^{(\nu+1/2)} - \frac{\Delta t}{\rho_{i-1/2,j}^n}\frac{(\delta p_{i,j}^{n+1})^{(\nu)}}{\Delta x}\\
(v_{i,j+1/2}^{n+1})^{(\nu+1/2)} &= (v_{i,j+1/2}^{n+1})^{(\nu)} + \frac{\Delta t}{\rho_{i,j+1/2}^n}\frac{(\delta p_{i,j}^{n+1})^{(\nu)}}{\Delta y}\\
(v_{i,j-1/2}^{n+1})^{(\nu+1)} &= (v_{i,j-1/2}^{n+1})^{(\nu+1/2)} - \frac{\Delta t}{\rho_{i,j-1/2}^n}\frac{(\delta p_{i,j}^{n+1})^{(\nu)}}{\Delta y}
\end{aligned}\right\} \tag{26}$$

The iterative procedure then is as follows. At each time level $n+1$, the iterative procedure is started by using the initial pressure iterates $(p_{i,j}^{n+1})^{(0)}$ equal to the pressure at the previous time level $p_{i,j}^n$. The initial velocity iterates are computed explicitly by use of equations (22) with $\nu = 0$. Then, for each cell (i,j), the pressure change $(\delta p_{i,j}^{n+1})^{(\nu)}$ is calculated with (24) and the pressure at the centre and the velocities on the sides of the cell (i,j) are updated by using (25) and (26), respectively. These operations are done by sweeping all the cells of the computational domain, from left to right and from bottom to top, until convergence has been achieved, that is, until, uniformly, $|(\delta p_{i,j}^{n+1})^{(\nu)}/(p_{i,j}^{n+1})^{(\nu+1)}| < \varepsilon$ for a fixed positive tolerance ε.

This iterative procedure for the pressure equations (18) is of practical implementation on digital computers and yields simultaneously the new fluid velocity $u_{i+1/2,j}^{n+1}$ and $v_{i,j+1/2}^{n+1}$ at each time step. The new values for the energy $E_{i,j}^{n+1}$ are then obtained from the equation of state (17).

VON KÁRMÁN VORTEX STREETS

Consider now a viscous, compressible fluid which flows in a rectangular domain defined by $-l < x < l$, $0 < y < h$. In order to account for the viscous effect on the flow, the viscosity terms must be included in the momentum equations of system (6) (see, e.g. Reference 6), but the viscous dissipation terms are neglected in the energy equation. Thus, the governing equations for this

problem are taken to be

$$\left.\begin{aligned}
&\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\mu}{\rho}\left(\frac{4}{3}\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{1}{3}\frac{\partial^2 v}{\partial x\partial y}\right)\\
&\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\frac{\mu}{\rho}\left(\frac{4}{3}\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial x^2}+\frac{1}{3}\frac{\partial^2 u}{\partial x\partial y}\right)\\
&\frac{\partial E}{\partial t}+u\frac{\partial E}{\partial x}+v\frac{\partial E}{\partial y}=-\gamma E\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\\
&\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial y}=-\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)
\end{aligned}\right\} \tag{27}$$

where the viscosity μ is assumed to be constant. The equation of state is again equation (7): $p=(\gamma-1)E$.

At the initial time $t_0=0$, the fluid is assumed to be flowing upward uniformly with the vertical boundaries and it is assumed to have uniform energy and density. Specifically, the initial conditions are taken to be

$$\left.\begin{aligned}
&u(x,y,0)=0\\
&v(x,y,0)=0{\cdot}16\\
&E(x,y,0)=1\\
&\rho(x,y,0)=1
\end{aligned}\right\} -l<x<l, 0<y<h \tag{28}$$

The velocity boundary conditions at the vertical boundaries are then

$$\left.\begin{aligned}
&u(-l,y,t)=u(l,y,t)=0\\
&v(-l,y,t)=v(l,y,t)=0{\cdot}16
\end{aligned}\right\} 0<y<h, t>0 \tag{29}$$

At the initial time $t_0=0$, a rigid obstacle is placed centrally along the inflow bottom boundary so that at subsequent times the flow is forced around the obstacle into the domain. Specifically, if the obstacle has width $2d$, with $d<l$, the velocity boundary conditions at the bottom horizontal boundary are taken to be

$$\left.\begin{aligned}
&u(x,0,t)=0, && -l<x<l, t>0,\\
&v(x,0,t)=0, && -d<x<d, t>0,\\
&v(x,0,t)=\frac{l}{l-d}0{\cdot}16, && -l<x<-d, d<x<l, t>0
\end{aligned}\right\} \tag{30}$$

These conditions ensure an entry flow rate equal to the initial flow rate when the obstacle was not present. The continuative outflow boundary conditions for the velocity are assumed at the top boundary to be

$$\left.\frac{\partial u}{\partial y}\right|_{y=h}=\left.\frac{\partial v}{\partial y}\right|_{y=h}=0, \quad -l<x<l, t>0 \tag{31}$$

At the inflow boundaries values for the energy and for the fluid density are given by

$$\left.\begin{aligned}
&E(x,0,t)=1\\
&\rho(x,0,t)=1
\end{aligned}\right\} -l<x<-d, d<x<l, t>0 \tag{32}$$

The parameters of the problem are taken to be $\mu = 0{\cdot}01$, $\gamma = 11$, $l = 20$, $h = 60$ and $d = 4$.

To avoid a symmetric flow pattern, we now simply change $u(17, 12{\cdot}75, 0)$ from 0 to 0·01.

Equations (27), with the given initial and boundary conditions, can now be solved using the numerical method described in the previous section. Note first that, owing to the presence of the viscous terms in the momentum equations of system (27), a simple space centred finite difference formula can be used to discretize the convective terms with no additional artificial viscosity required for stability. The viscous terms can also be discretized with simple, explicit finite differences, which are then included in $F^n_{i+1/2,j}$ and $G^n_{i,j+1/2}$. Specifically, $F^n_{i+1/2,j}$, $G^n_{i,j+1/2}$ and $H^n_{i,j}$, in this problem, are taken to be

$$F^n_{i+1/2,j} = u^n_{i+1/2,j}\frac{u^n_{i+3/2,j} - u^n_{i-1/2,j}}{2\Delta x} + v^n_{i+1/2,j}\frac{u^n_{i+1/2,j+1} - u^n_{i+1/2,j-1}}{2\Delta y}$$
$$- \frac{\mu}{\rho^n_{i+1/2,j}}\left[\frac{4u^n_{i+3/2,j} - 2u^n_{i+1/2,j} + u^n_{i-1/2,j}}{3(\Delta x)^2}\right.$$
$$+ \frac{u^n_{i+1/2,j+1} - 2u^n_{i+1/2,j} + u^n_{i+1/2,j-1}}{(\Delta y)^2}$$
$$\left. + \frac{1}{3}\frac{v^n_{i+1,j+1/2} - v^n_{i+1,j-1/2} - v^n_{i,j+1/2} + v^n_{i,j-1/2}}{(\Delta x)(\Delta y)}\right]$$

$$G^n_{i,j+1/2} = u^n_{i,j+1/2}\frac{v^n_{i+1,j+1/2} - v^n_{i-1,j+1/2}}{2\Delta x} + v^n_{i,j+1/2}\frac{v^n_{i,j+3/2} - v^n_{i,j-1/2}}{2\Delta y}$$
$$- \frac{\mu}{\rho^n_{i,j+1/2}}\left[\frac{4v^n_{i,j+3/2} - 2v^n_{i,j+1/2} + v^n_{i,j-1/2}}{3(\Delta y)^2} + \frac{v^n_{i+1,j+1/2} - 2v^n_{i,j+1/2} + v^n_{i-1,j+1/2}}{(\Delta x)^2}\right.$$
$$\left. + \frac{1}{3}\frac{u^n_{i+1/2,j+1} - u^n_{i+1/2,j} - u^n_{i-1/2,j+1} + u^n_{i-1/2,j}}{(\Delta x)(\Delta y)}\right] \tag{34}$$

$$H^n_{i,j} = u^n_{i,j}\frac{E^n_{i+1,j} - E^n_{i-1,j}}{2\Delta x} + v^n_{i,j}\frac{E^n_{i,j+1} - E^n_{i,j-1}}{2\Delta y} \tag{35}$$

where the discrete variables $u^n_{i,j+1/2}$, $v^n_{i+1/2,j}$, $\rho^n_{i+1/2,j}$, $\rho^n_{i,j+1/2}$, $u^n_{i,j}$ and $v^n_{i,j}$ are defined as simple averages from the closest scalar grid points.[1]

The flow domain is now covered with 40 by 40 finite difference cells having width $\Delta x = 1$ and height $\Delta y = 1{\cdot}5$, the overrelaxation factor is fixed to be $\omega = 1{\cdot}5$, the convergence criterion ε is 10^{-3} and a numerical solution is generated at times $t_n = n\Delta t$, $n = 1, 2, \ldots$, with a time increment $\Delta t = 1$. Figures 2–7 show the computer generated velocity field relative to the moving vertical boundaries at times $t = 500$, 1000, 1500, 2000, 2500 and 3000, respectively. Initially, the wake of the obstacle is laminar; then, as time advances, a regular pattern forms of vortices which move alternately clockwise and counterclockwise. At large times the solution becomes periodic with a period of approximately 200 time units. Figures 5 and 7 show the solution after five periods, and they are, in fact, almost entirely identical; whereas Figures 6 and 7 show the solution after two and half periods, and they are almost entirely symmetrical. These vortex patterns form the well-known von Kármán vortex street, which is frequently observed in experiments when the Reynolds number is between certain limits.[11] The deformation of the uppermost vortex shown in Figures 5–7 is due to its interaction with the lateral walls. Note, finally, that the speed of sound at the inflow boundaries is $a = \sqrt{110}$ and hence our method has allowed the use of a time step which is over ten times the maximum time step allowed by the Courant condition.

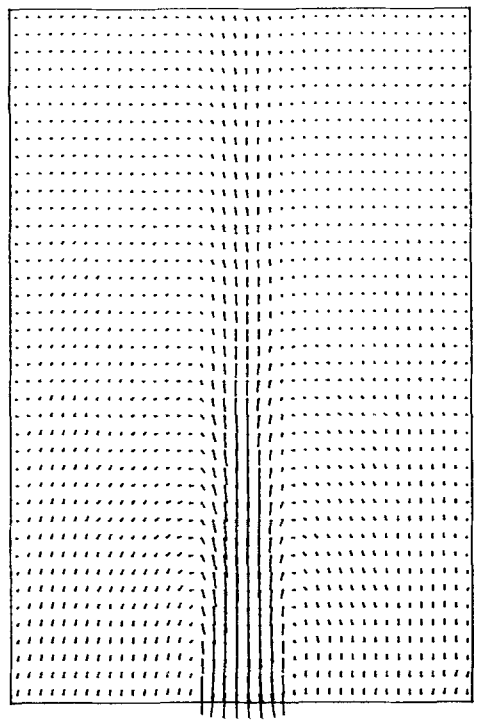

Figure 2. $t = 500$

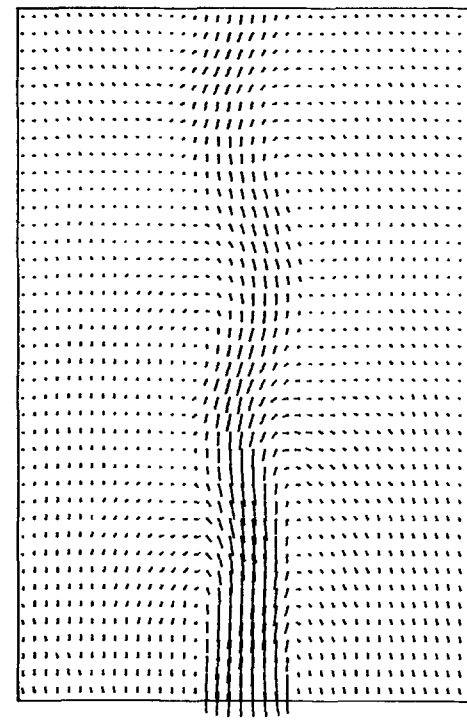

Figure 3. $t = 1000$

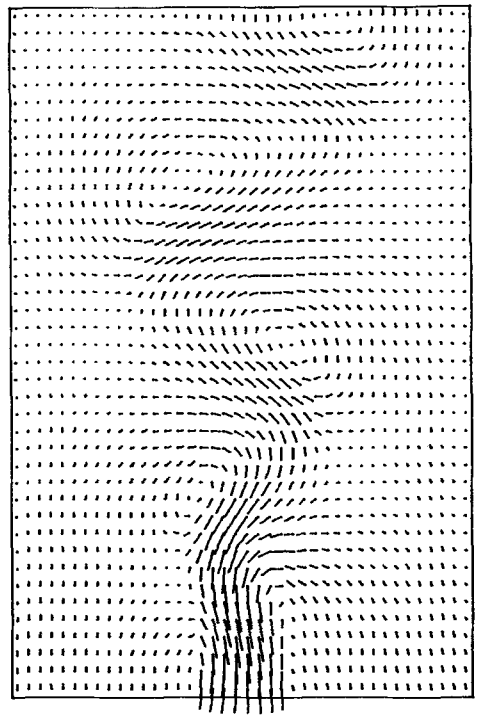

Figure 4. $t = 1500$

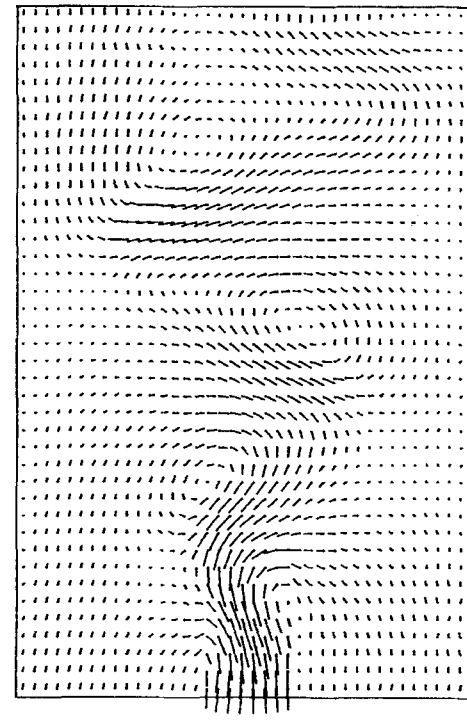

Figure 5. $t = 2000$

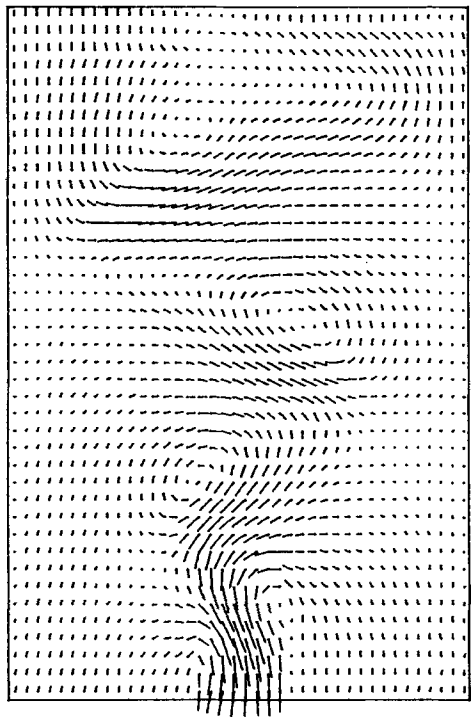

Figure 6. $t = 2500$

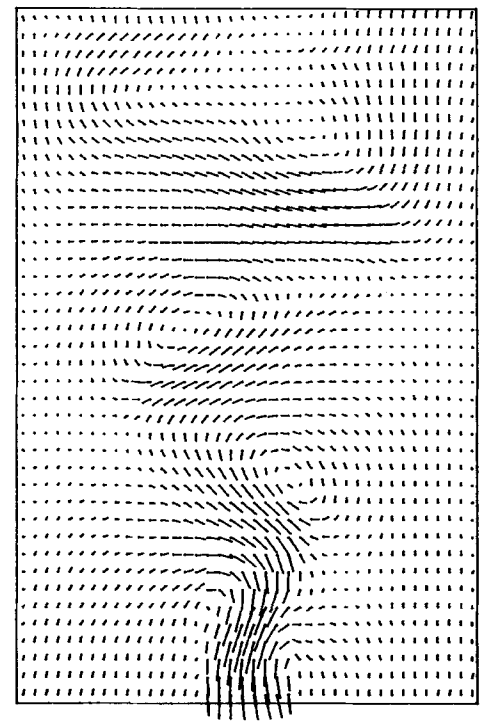

Figure 7. $t = 3000$

These computations were performed on the DEC-20 computer at The University of Texas at Arlington and the solution at $t = 3000$ was reached in only 45 min. of CPU time. For the interested reader, the FORTRAN program for this specific problem is provided in the Appendix of Reference 1.2.

CONCLUSION

The method presented in this paper is a new finite difference method for transient, compressible fluid flow simulation. The characteristic equation of the flow model is first studied in order to determine that special semi-implicit discretization which allows us to avoid the Courant stability condition for the speed of sound. An efficient algorithm for the practical implementation of this method on digital computers is also provided. As an example, a von Kármán vortex street has been simulated and the results thus obtained compare well with observation and experimentation. Use of this method for other compressible fluid flow problems is deemed promising.

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